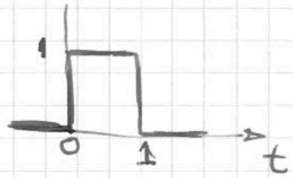
DE2.3 Electronics 2 for Design Engineers

Tutorial Sheet 3 – Laplace Transform and Transfer Function

SOLUTIONS

1.*

a) $f(t) = u(t) - u(t-1)$



$$F(s) = \int_0^1 e^{-st}\,dt$$

$$= -\frac{e^{-st}}{s}\Big|_0^1$$

$$= -\frac{1}{s}\left(e^{-s} - 1\right)$$

$$= \frac{1}{s}\left(1 - e^{-s}\right)$$

b) $f(t) = te^{-t}u(t)$

$$F(s) = \int_0^\infty t e^{-t} e^{-st}\,dt = \int_0^\infty t e^{-(s+1)t}\,dt$$

$\boxed{\text{Use integration by parts} \Longrightarrow \quad f(t) = t, \quad g'(t) = e^{-(s+1)t}}$

$$= -\frac{e^{-(s+1)t}}{(s+1)^2}\left[-(s+1)t - 1\right]\Big|_0^\infty$$

In order to guarantee convergence, we need

$e^{-(s+1)t} \to 0$ as $t \to \infty$, or $Re(s+1) > 0$.

Then

$$F(s) = \underbrace{\frac{e^{-(s+1)t}}{(s+1)^2}(s+1)t\Big|_0^\infty}_{=\,0} + \frac{e^{-(s+1)t}}{(s+1)^2}\Big|_0^\infty$$

$$= \frac{1}{(s+1)^2}$$

(Note: This question is actually quite difficult!).

c) $f(t) = t \cos \omega_0 t \, u(t)$

$$F(s) = \int_0^\infty t \cos \omega_0 t \, e^{-st} \, dt$$

$$= \frac{1}{2}\left\{\int_0^\infty \left[t e^{(j\omega_0 - s)t} + t e^{-(j\omega_0 + s)t}\right] dt\right\}$$

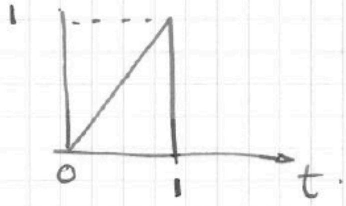
$$= \frac{1}{2}\left[\frac{1}{(s - j\omega_0)^2} + \frac{1}{(s + j\omega_0)^2}\right] \quad Re(s) > 0$$

$$= \frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2} \quad //$$

I will NOT expect you to remember formulae except the basic definition of Laplace transform.

2.

a)



$$F(s) = \int_0^1 t e^{-st}\, dt$$

$$= -\frac{t}{s} e^{-st}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\, dt$$

$$= -\frac{t}{s} e^{-st}\Big|_0^1 - \frac{1}{s^2} e^{-st}\Big|_0^1$$

$$= -\frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} + \frac{1}{s}$$

$$= \frac{1}{s^2}\left(1 - e^{-s} - s e^{-s}\right) \quad //$$

Integration by parts

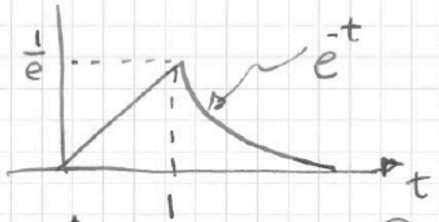
$$\int_a^b f(t) g'(t)\, dt$$

$$= f(t) g(t)\Big|_a^b - \int_a^b f'(t) g(t)\, dt$$

Let $g'(t) = e^{-st}$

$g(t) = -\frac{1}{s} e^{-st}$ $\quad f'(t) = 1$

b)



$$F(s) = \int_0^1 \frac{t}{e} e^{-st}\, dt + \int_1^{\infty} e^{-t} e^{-st}\, dt$$

$$= \frac{1}{e}\int t e^{-st} dt + \int_1^{\infty} e^{-(s+1)t}\, dt$$

$$= \frac{e^{-st}}{es}\left(-st - 1\right)\Big|_0^1 - \frac{1}{s+1} e^{-(s+1)t}\Big|_1^{\infty}$$

$$= \underbrace{\frac{1}{es^2}\left(1 - e^{-s} - s e^{-s}\right)}_{\text{similar to Q2a)}} + \frac{1}{s+1} e^{-(s+1)} \quad //$$

3. This year, I did not really cover the topic of inverse Laplace transform. So this question will NOT be examinable.

a) $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+5s+6}\right\}$

$$\frac{2s+5}{s^2+5s+6} = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3}$$

$$\therefore\ f(t) = \left(e^{-2t} + e^{-3t}\right)u(t) \quad //$$

b)

$$F(s) = \frac{(s+1)^2}{s^2-s-6} = \frac{(s+1)^2}{(s+2)(s-3)}$$

Since the order of numerator = order of denominator, this is an improper fraction.

An example is given in lecture 6 slide 12.

Using method provided in lecture,

$$F(s) = 1 + \frac{a}{s+2} + \frac{b}{s-3} = 1 - \frac{0.2}{s+2} + \frac{3.2}{s-3}$$

This is the coefficient of the s^2 term in numerator.

$$\therefore\ f(t) = \delta(t) + \left(3.2e^{3t} - 0.2e^{-2t}\right)u(t)$$

4. a)

$$f(t) = u(t) - u(t-1)$$

$$\therefore \quad F(s) = \mathcal{L}[u(t)] - \mathcal{L}[u(t-1)]$$

$$= \frac{1}{s} - e^{-s}\frac{1}{s} = \frac{1}{s}(1 - e^{-s}) \quad //$$

$$\boxed{\text{Important: Compare this solution with that of Q1 a), this is much easier.}}$$

b)

$$f(t) = e^{-(t-\tau)}u(t) = e^{\tau}\, e^{-t}u(t).$$

$$\therefore \quad F(s) = e^{\tau}\frac{1}{s+1} \quad //$$

c)

$$f(t) = \sin \omega_0(t-\tau)\, u(t-\tau).$$

This is $\sin \omega_0 t$ delayed by τ.

$$\therefore \quad F(s) = \left(\frac{\omega_0}{s^2 + \omega_0^2}\right) e^{-s\tau} \quad //$$

5. a)

$\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 24y(t) = 5\frac{df}{dt} + 3f(t)$

Take Laplace transform on both sides:

$(s^2 + 11s + 24)Y(s) = (5s + 3)F(s)$

Transfer function $H(s) = \frac{Y(s)}{F(s)} = \frac{5s+3}{s^2+11s+24}$

b)

$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} - 11\frac{dy}{dt} + 6y(t) = 3\frac{d^2f}{dt^2} + 7\frac{df}{dt} + 5f(t)$

$(s^3 + 6s^2 - 11s + 6)Y(s) = (3s^2 + 7s + 5)F(s)$

$H(s) = \frac{Y(s)}{F(s)} = \frac{3s^2+7s+5}{6s^2-11s+6}$